Example 0:

$$\int \frac{1}{x((\ln(x))^2+81)}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$\int \frac{1}{u^2+81}\,du$$

$$= \int \frac{1}{81\left(\frac{u^2}{81}+1\right)}\,du$$

$$\int \frac{1}{81\left(\frac{u^2}{81}+1\right)} du$$

$$= \frac{1}{81} \int \frac{1}{\left(\frac{u^2}{81}+1\right)} du$$

$$= \frac{1}{81} \int \frac{1}{\left(\frac{u}{9}\right)^2+1} du$$

$$s = \frac{u}{9}$$

$$ds = \frac{1}{9} du, \quad \text{so } 9\,ds = du$$

$$= \frac{1}{81} \int \frac{9\,ds}{s^2+1}$$

$$\frac{1}{81}\int \frac{9\,ds}{s^2+1}$$

$$= \frac{1}{9}\int \frac{ds}{s^2+1}$$

$$= \frac{1}{9}\arctan(s) + C$$

$$= \frac{1}{9}\arctan\left(\frac{u}{9}\right) + C$$

$$= \boxed{\frac{1}{9}\arctan\left(\frac{\ln(x)}{9}\right) + C}$$

Example 1: $\int \cos^3(x)\ln(\sin(x))\,dx$

$u = \sin(x)$

$\boxed{du = \cos(x)\,dx}$

$= \int \cos^2(x)\ln(\sin(x))\boxed{\cos(x)\,dx}$

$= \int (1-\sin^2(x))\ln(\sin(x))\cos(x)\,dx$

$= \int (1-u^2)\ln(u)\,du$

$$\int (1-u^2)\ln(u)\,du$$

$$w = \ln(u) \qquad v = u - \frac{u^3}{3}$$

$$dw = \frac{1}{u}\,du \qquad dv = (1-u^2)\,du$$

$$= \left(u - \frac{u^3}{3}\right)\ln(u) - \int \left(u - \frac{u^3}{3}\right)\cdot\frac{1}{u}\,du$$

$$= \left(u - \frac{u^3}{3}\right)\ln(u) - \int \left(1 - \frac{u^2}{3}\right)du$$

$$= \left(u - \frac{u^3}{3}\right)\ln(u) - \left(u - \frac{u^3}{9}\right) + C$$

$$= \boxed{\left(\sin(x) - \frac{\sin^3(x)}{3}\right)\ln(\sin(x)) - \sin(x) + \frac{\sin^3(x)}{9} + C}$$

In general:

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx, \quad n \neq m$$

use trig identity

$$\cos(A+B) = \cancel{\cos(A)\cos(B)} - \sin(A)\sin(B)$$

$$-\left(\cos(A-B) = \cancel{\cos(A)\cos(B)} + \sin(A)\sin(B)\right)$$

Subtract to get

$$\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B)$$

so $$\boxed{\sin(A)\sin(B) = \frac{\cos(A-B) - \cos(A+B)}{2}}$$

Using this identity,

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \quad \begin{pmatrix} A = nx \\ B = mx \end{pmatrix}$$

$$= \int_{-\pi}^{\pi} \frac{\cos(nx-mx) - \cos(nx+mx)}{2}\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos((n-m)x) - \cos((n+m)x)\right)dx$$

$$= \frac{1}{2}\left(\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m}\right)\Bigg|_{-\pi}^{\pi}$$

$= \boxed{0}$ since $\sin(k\pi) = 0$ for all integers k.

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx, \quad n \neq m$$

use the same trig identity, except add instead of subtract

You'll get that

$$\boxed{\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

$u = \cos(nx)$

$du = -\sin(nx)\,dx$

(if $m=n$, this works)

$u(\pi) = \cos(n\pi) = (-1)^n$

$u(\pi) = (-1)^n$

$$-\int_{(-1)^n}^{(-1)^n} u\,du = 0$$

$m \neq n$, use trig identity

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

again, we'd get zero.

You will not be tested on any trig identity used on the last three integrals!

Other trig Integrals

Example 2:

$$\int_0^{\pi/4} \sin^3(x)\cos^2(x)\,dx$$

$\sin^2(x)\sin(x)$

$$\sin^2(x)+\cos^2(x)=1$$

so $\sin^2(x) = 1-\cos^2(x)$

$$\int_0^{\pi/4} \sin(x)\cdot\cos^2(x)\cdot\sin^2(x)\,dx$$

$$= \int_0^{\pi/4} \sin(x)\cdot\cos^2(x)(1-\cos^2(x))\,dx$$

$$\int_0^{\pi/4} \boxed{\sin(x)} \cdot \cos^2(x)\left(1-\cos^2(x)\right) dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$, so

$-du = \boxed{\sin(x)\,dx}$

$u(0) = 1$

$u\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

The integral becomes

$$-\int_1^{\sqrt{2}/2} u^2(1-u^2)\,du$$

$$= \int_{\sqrt{2}/2}^{1} (u^2 - u^4)\,du$$

$$\int_{\sqrt{2}/2}^{1} \left(u^2 - u^4\right) du$$

$$= \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Bigg|_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \boxed{\left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{40}\right)}$$

Example 3:

$$\int \frac{\cos^7(e^{-x})\sin^4(e^{-x})}{e^x}\,dx$$

$$u = e^{-x}$$

$$du = -e^{-x}\,dx, \quad -du = \boxed{e^{-x}dx}$$

$$\Rightarrow \int \cos^7(e^{-x})\sin^4(e^{-x})\boxed{e^{-x}dx}$$

$$\left(1/e^x = e^{-x}\right)$$

$$= -\int \cos^7(u)\sin^4(u)\,du$$

$$-\int \cos^7(u)\sin^4(u)\,du$$

$$= -\int \cos(u)\cos^6(u)\sin^4(u)\,du$$

$$= -\int \boxed{\cos(u)\,\boxed{(\cos^2(u))^3\sin^4(u)}\,du}$$

$$w = \sin(u)$$

$$dw = \boxed{\cos(u)\,du}$$

use trig identity $1-\sin^2(u) = \cos^2(u)$

$$= -\int \cos(u)(1-\sin^2(u))^3\sin^4(u)\,du$$

now substitute to get

$$= -\int (1-w^2)^3\,w^4\,dw$$

$$-\int (1-w^2)^3 \, w^4 \, dw$$

$$= -\int (1-3w^2+3w^4-w^6)\, w^4 \, dw$$

$$= -\int (w^4-3w^6+3w^8-w^{10})\, dw$$

$$= -\left(\frac{w^5}{5}-\frac{3w^7}{7}+\frac{3w^9}{9}-\frac{w^{11}}{11}\right)+C$$

$$= -\left(\frac{\sin^5(u)}{5}-\frac{3\sin^7(u)}{7}+\frac{3\sin^9(u)}{9}-\frac{\sin^{11}(u)}{11}\right)+C$$

$$\boxed{= -\left(\frac{\sin^5(e^{-x})}{5}-\frac{3\sin^7(e^{-x})}{7}+\frac{3\sin^9(e^{-x})}{9}-\frac{\sin^{11}(e^{-x})}{11}\right)+C}$$

Example 4:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(x)\sin^4(x)\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2(x)\right)^2 \left(\sin^2(x)\right)^2 dx$$

use

$$\left.\begin{aligned} \cos^2(x) &= \frac{1+\cos(2x)}{2} \\ \sin^2(x) &= \frac{1-\cos(2x)}{2} \end{aligned}\right\} \text{Know these!}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2(x)\sin^2(x)\cos^2(x)\sin^2(x)\right) dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2(x)\sin^2(x)\cos^2(x)\sin^2(x)\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1-\cos^2(2x)\right)\left(1-\cos^2(2x)\right)dx$$

$$= \frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2x)\cdot\sin^2(2x)\,dx$$

$$\frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2x)\cdot\sin^2(2x)\,dx$$

$\sin^2(2x) = \frac{1-\cos(4x)}{2}$, so we get

$$\frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1-\cos(4x)}{2}\right)\left(\frac{1-\cos(4x)}{2}\right)dx$$

$$= \frac{1}{64}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1-2\cos(4x)+\cos^2(4x)\right)dx$$

$$\cos^2(4x) = \frac{1+\cos(8x)}{2}$$

$$\frac{1}{64}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-2\cos(4x)+\cos^2(4x)\right)dx$$

$$=\frac{1}{64}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-2\cos(4x)+\left(\frac{1+\cos(8x)}{2}\right)\right)dx$$

$$=\frac{1}{64}\left(x-\cancel{\frac{\sin(4x)}{2}}+\frac{x}{2}+\cancel{\frac{\sin(8x)}{16}}\;\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

vanish when you plug in bounds

$$=\boxed{\frac{1}{64}\;\frac{3\pi}{2}}$$

$$\int \cos^m(x)\sin^n(x)\,dx$$, m, n counting numbers

1) m odd factor out a single cos, change everthing else to sines, $u = \sin(x)$,

2) n odd factor out a single sin, change everything else to cosines, $u = \cos(x)$

3) n, m even use double angle formulas

Trig Substitution Section 7.3

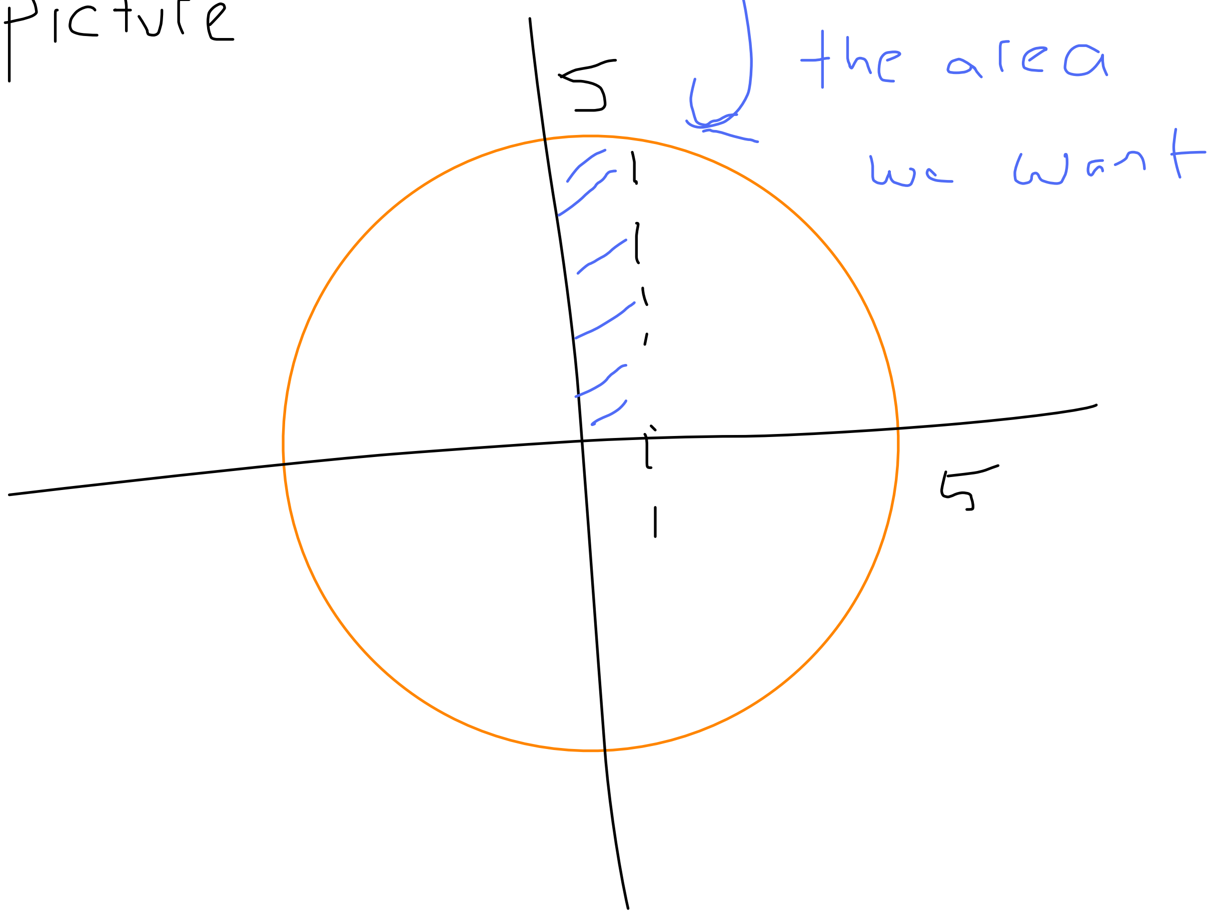
$$\int_{-5}^{5} \sqrt{25 - x^2}\, dx = \frac{25\pi}{2}$$

by using geometry

What about

$$\boxed{\int_0^1 \sqrt{25-x^2}\,dx\ ?}$$

Picture



Use the substitution

$$u = \arcsin\left(\frac{x}{5}\right)$$

why?

Because if you take sine of either side, you get

$$\sin(u) = \sin\left(\arcsin\frac{x}{5}\right) = \frac{x}{5}$$

so $x = 5\sin(u)$

$$dx = 5\cos(u)\,du$$

$$x = \boxed{5\sin(u)}$$

$$dx = 5\cos(u)\,du$$

$$\int_0^1 \sqrt{25 - \boxed{x}^2}\,\boxed{dx}$$

take bounds off!

$$= \int \sqrt{25 - 25\sin^2(u)}\; 5\cos(u)\,du$$

$$= \int 5\sqrt{1 - \sin^2(u)}\; 5\cos(u)\,du$$

$$= 25\int \sqrt{\cos^2(u)}\; \cos(u)\,du$$

This is

$$25 \int \cos^2(u)\, du$$

$$= 25 \int \frac{1+\cos(2u)}{2}\, du$$

$$= 25 \left(\frac{u}{2} + \frac{\sin(2u)}{4} \right) \Bigg|$$ bounds?

$x=0$ to $x=1$

$u = \arcsin\left(\frac{x}{5}\right)$ when $x=0$, $u=0$

when $x=1$, $\underbrace{u = \arcsin\left(\frac{1}{5}\right)}_{?}$

$$25\left(\frac{u}{2}+\frac{\sin(2u)}{4}\right)\Bigg|_0^{\arcsin(\frac{1}{5})}$$

$$=25\left(\frac{\arcsin(\frac{1}{5})}{2}+\frac{\sin(2\arcsin(\frac{1}{5}))}{4}\right)$$

$\sin 2\theta = 2\sin\theta\cos\theta$, so

$$\sin(2\arcsin(\tfrac{1}{5})) = 2\sin(\arcsin(\tfrac{1}{5}))\cdot\cos(\arcsin(\tfrac{1}{5}))$$

$$=\frac{2}{5}\cdot\frac{\sqrt{24}}{5}=\frac{2\sqrt{24}}{25}$$

$$=\boxed{25\left(\frac{\arcsin(\frac{1}{5})}{2}+\frac{2\sqrt{24}}{100}\right)}$$